Thursday, April 06, 2017
8:20 AM

As an example consider a constant density star (analogous to our E+M example).

$$\rho(r) = \begin{cases} \rho_* & r \le R \\ 0 & r > R \end{cases}$$

a) $$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_* & r \le R \\ \frac{4}{3}\pi R^3 \rho_* = M & r > R \end{cases} \Rightarrow e^{2\beta(r)} = \left[1 - \frac{8\pi G \rho_* r^2}{3}\right]^{-1}$$

b) $$\frac{dp}{dr} = -\frac{(\rho_* + p)\left[G \frac{4}{3}\pi r^3 \rho_* + 4\pi G r^3 p\right]}{r\left[r - 2G\frac{4}{3}\pi r^3 \rho_*\right]}$$

$\Downarrow$ after a "little" work

$$p(r) = \rho_* \left\{ \frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R-2GM}} \right\} \qquad r < R$$

c) Solving for $\alpha(r)$ we get: $e^{\alpha(r)} = \frac{3}{2}\left(1 - \frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{R^3}\right)^{1/2} \qquad r < R$

Note: @ $r = R$ $e^{\alpha(R)} = \left(1 - \frac{2GM}{R}\right)^{1/2} \Rightarrow e^{2\alpha(R)} = \left(1 - \frac{2GM}{R}\right)$ agreeing w/ Schwarzchild!
We already know that $e^{2\beta(r)}$ matches.

Consider the $p(r)$ expression:

- $p(r)$ increases as r decreases (makes sense!)
- The pressure at $r = 0$ blows up as $M \to \frac{4}{9}\frac{R}{G}$ (from below), that is $R^{3/2} - 3R\sqrt{R - 2GM} \to 0$.
- Therefore if $M > \frac{4}{9}\frac{R}{G}$ then this solution is inconsistent.
  The consistency failure comes from our static assumption.
  If $M > \frac{4}{9}\frac{R}{G}$ the system will evolve w/ time!

Thursday, April 06, 2017
8:33 AM

What will happen?
Assuming spherical symmetry, the system will collapse.

Essentially ρ is responsible for the gravitational attraction and p provides the outward pressure. In a static scenario these balance, but for $M > \frac{4}{9}\frac{R}{G}$ the pressure loses.

Important: As it collapses the density increases and the condition $M > \frac{4}{9}\frac{R}{G}$ is maintained since M is constant and R decreases. So the collapse in this scenario does not end and we are left with a black hole.

The Real World

2 issues need to be addressed before accepting this story:

a) $\rho(r) = \text{constant}$
Although this is actually a pretty good model for stellar objects, it can be shown that for more general $\rho(r)$ and spherical symmetry, the condition $M > \frac{4}{9}\frac{R}{G}$ still leads to collapse (Buchdahl's Theorem).

b) More pressing is that for our sun: $(c^2)\,\frac{4}{9}\frac{R_\odot}{G} = 1.38\times 10^{27}\,\text{kg}$, $M_\odot = 1.98\times 10^{30}\,\text{kg}$ $\Big\}\; M_\odot \gg \frac{4}{9}\frac{R_\odot}{G}$

So why doesn't our sun collapse?!

Thursday, April 06, 2017
8:41 AM

Our analysis was based on a perfect fluid model, i.e. non-interacting particles. Obviously stellar interiors are undergoing (among others) nuclear interactions which generate an outward pressure. Taking this into account, the gravitational pressure does not have to be as large in order to balance the gravitational attraction. Until...

When the stars nuclear fuel burns out it will begin to collapse.

At a certain point in the collapse the electron-degeneracy pressure can become large enough to halt collapse and we are left w/ a white dwarf.

Chandrasekhar found that for $M > 1.4 M_\odot$, even the electron degeneracy pressure cannot halt collapse.

Eventually electrons and protons fuse to create neutrons and the neutron degeneracy pressure can halt collapse leaving a neutron star (of radius ~10km).

However if the Oppenheimer-Volkoff limit of $3 \sim 4 M_\odot$ is exceeded, even the neutron degeneracy pressure can't halt collapse and we will be left with a black hole.

Tuesday, March 24, 2015
6:57 PM

Schwarzschild Black Holes

$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$ - spherically symmetric solution to $R_{\mu\nu}=0$

Clearly, this gets interesting for $r=2GM$ and $r=0$.
For "normal" astrophysical objects, e.g. stars, planets, and indeed for any spherical object like a basketball or a cow, both $r=2GM$ and $r=0$ are inside of the object. But recall that for extended sources this solution only holds outside. For the interior we must solve EE w/ $T_{\mu\nu}\neq 0$.

However we learned last time that objects of mass $M > 3\sim 4 M_\odot$ will eventually collapse to form a black hole. In this case $r=2GM$ is well outside of the interior, and in fact we can approach $r\to 0$ as close as we like (or are dumb enough to do) and still be "outside" of the source.

Black holes sometimes get a bad wrap. But it is important to recall that the Schwarzschild solution describes the exterior geometry of any spherically symmetric source. So black holes do not "suck" any harder than comparable mass stars (as long as we stay "outside"). Of course falling into a black sucks, but so does falling into a star.

What makes black holes so interesting is that we don't hit the interior until $r=0$, yet the geometry we encounter along the way does some really interesting stuff.

Before continuing, let's review a silly idea.

Newtonian Gravity: Escape velocity - the kinetic energy needed to escape to infinity from a gravitating body.

$$E_{tot} = \frac{1}{2} m v_{escape}^2 - \frac{GMm}{r} = 0$$

Note: $v \to 0$ as $r \to \infty$, i.e. it just barely escapes!

$$\Downarrow$$

$$v_{escape} = \sqrt{\frac{2GM}{R}}$$

But observe that if $R = 2GM$ then $v_{escape} = 1\ (c)$, so no massive object can completely escape to $r = \infty$.

Is this a black hole? Nope!

2 problems w/ trying to identify this as a black hole:

a) This analysis ignores the possibility of doing anything else to escape. There is nothing that prevents us from using a thruster of some sort to escape from an object w/ $R = 2GM$. However, we will find that for true black holes, once you are inside of $R = 2GM$, there is absolutely nothing you can do to escape.

b) In this analysis, even though you can't escape, you can still move away from $r = 0$. For true black holes, once inside of $R = 2GM$, the only direction you move is toward $r = 0$.

Back to General Relativity:

Clearly something interesting happens to ds^2 when $r = 2GM$. It looks singular from $(1-\frac{2GM}{r})^{-1}dr^2$ blowing up. But does this mean that something in the geometry is becoming singular at $r = 2GM$? Is the curvature blowing up?

The answer is no. To help understand consider $\mathbb{R}^2$ w/ $\{r,\theta\}$: $g_{ij} = \begin{pmatrix}1 & \\ & r^2\end{pmatrix} \Rightarrow g^{ij} = \begin{pmatrix}1 & \\ & r^{-2}\end{pmatrix}$

Notice that for $r \to 0$ g_{ij} is degenerate and g^{ij} is singular. But we already know in this case that nothing strange is happening at $r = 0$ since this is good old $\mathbb{R}^2$!

To systematically see and handle this:

a) We should remember that the metric is coordinate dependent. But we should look for invariant statements about the geometry. For this space $R^i{}_{jkl} = 0$, and of course all curvature invariants we could imagine building from this, e.g. R, $R_{ij}R^{ij}$, $R^i{}_{jkl}R_i{}^{jkl}$, etc. are also 0. Lesson: To check for funky geometry, explore curvature invariants.

b) We know that $r = 0$ is better handled w/ a different choice of coordinates, e.g. $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\} \Rightarrow \underbrace{g_{ij} = \begin{pmatrix}1 & \\ & 1\end{pmatrix} \Rightarrow g^{ij} = \begin{pmatrix}1 & \\ & 1\end{pmatrix}}_{\text{No problems at } x=y=0 \text{ (i.e. } r=0\text{)!}}$

Lesson: When things look bad, but we know they aren't (from curvature invariants) it often pays to explore "better" coordinate options.

In this example ($\mathbb{R}^2$ w/ $\{r,\theta\}$) $r = 0$ is a coordinate singularity. This means the metric, in this set of coordinates is doing something funny, but there is nothing singular happening w/ the curvature.

Getting back to our Schwarzschild metric we can apply the lesson from (a) to the $r = 2GM$ and $r = 0$ suspects. First recall that $R_{\mu\nu} = 0 \Rightarrow R = 0$, so we need to go back to $R^\mu{}_{\nu\rho\lambda}$. In fact we find $\underbrace{R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}}= \frac{48G^2M^2}{r^6}$ and all other invariants are finite. But while we notice that $\Rightarrow \infty$ for $r \to 0$ (which we expect) it is finite for $r = 2GM$.

So while $r = 0$ represents a true curvature singularity, $r = 2GM$ is only a coordinate singularity (albeit a very interesting one). So now we turn to the lesson from (b), i.e. when you encounter a coordinate singularity, look for better coordinates.

Wednesday, April 12, 2017
9:30 AM

Better coordinates:

$$(t, r, \theta, \phi)_{\text{Schwarzschild}} \longrightarrow (v, r, \theta, \phi)_{\text{Eddington-Finkelstein}}$$

where $v = t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right|$

or $t = v - r - 2GM \ln\left|\frac{r}{2GM} - 1\right| \quad \Rightarrow dt = dv - \left(1 - \frac{2GM}{r}\right)^{-1} dr$

Then $ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$ "The Schwarzschild geometry in Eddington-Finkelstein coordinates"

Note:

- $r = 2GM$ is no longer problematic
- $r = 0$ is still problematic, but we should expect this since it is a true curvature singularity.
- Schwarzschild coordinates go haywire near $r = 2GM$, but are otherwise useful for $r > 2GM$ and $r < 2GM$. EF coordinates on the other hand are useful everywhere except @ $r = 0$. In particular they are more reliable for describing things as they pass through $r = 2GM$.

Thursday, April 13, 2017
8:52 AM

To get at some of the weirdness of BH geometries, it helps to examine the causal structure, i.e. light cones.

Recall that light travels along null paths, i.e. $ds^2 = 0$.
Let's focus on purely radial ($d\theta = d\phi = 0$) trajectories for simplicity.

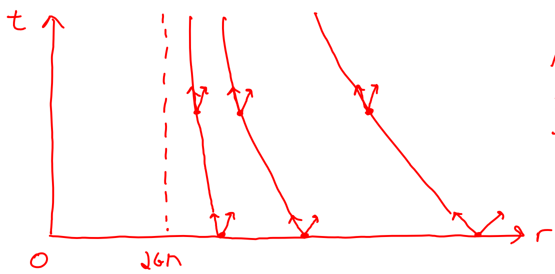
- Schwarzschild $\quad 0 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2$
- Eddington-Finkelstein $\quad 0 = -\left(1-\frac{2GM}{r}\right)dv^2 + 2\,dv\,dr$

To get a sense of what is going on we can draw the behavior of the light-cones in each case on a spacetime diagram.

Schwarzschild: $\quad 0 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2$

$$\frac{dr}{dt} = \pm\left(1-\frac{2GM}{r}\right) \longrightarrow \begin{cases} \pm 1 \text{ as } r\to\infty & \text{propagates at } c \text{ far from } 2GM \\ 0 \text{ as } r\to 2GM & \text{slows to rest at } 2GM \end{cases}$$

To draw let's use $\frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1} \Rightarrow t = \pm\left(r + 2GM\ln\left[\frac{r}{2GM}-1\right]\right) + \text{constant}$



Notice how the light-cones are at the usual ± 1 for $r\to\infty$, but "close-up" as $r\to 2GM$. This closing up of the light-cones is a visual signal of the problem w/ Schwarzschild coordinates as $r\to 2GM$. The situation should improve w/ Eddington-Finkelstein.

Thursday, April 13, 2017
9:07 AM

Eddington-Finkelstein: $0 = -(1-\frac{2GM}{r})dv^2 + 2dvdr$

This time we get several possible null trajectories.

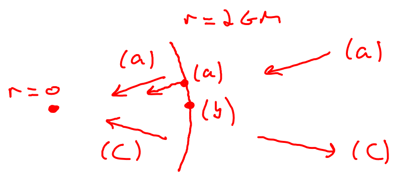
a) $dv = 0 \Rightarrow v = \text{constant} = \underline{t + r + 2GM\ln|\frac{r}{2GM}-1|}$

as t increases r must decrease so these must describe "ingoing" trajectories (for $r \gtrless 2GM$)

b) $dr = 0$ and $r = 2GM$ radially stationary, i.e. neither ingoing or outgoing

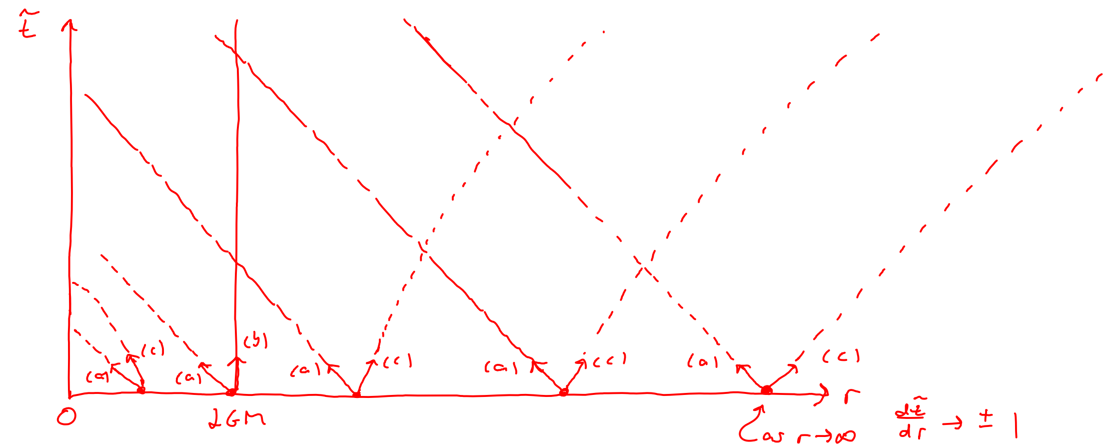
c) $\frac{dr}{dv} = \frac{1}{2}(1-\frac{2GM}{r}) = \begin{cases} >0 & r>2GM & \text{outgoing} \\ <0 & r<2GM & \text{ingoing} \end{cases}$

Summarizing:



Note: For $r<2GM$, only ingoing trajectories exist. At $r=2GM$ there is one stationary trajectory.

To graph the light-cones, we first utilize a Schwarzschild compatible time coordinate $\tilde{t} = v - r$ for which $\tilde{t} \to t$ as $r \to \infty$.



as $r \to \infty$ $\frac{d\tilde{t}}{dr} \to \pm 1$

So in Eddington-Finkelstein coordinates (which are useful for passing through $r=2GM$) we notice that the light-cones "tip over". But recall that all motion is restricted to the future light-cone, so we see why for $r<2GM$ only ingoing trajectories exist.